Chapter 5
Probability

In this chapter, we look at basic properties of probability.

5.1 Probability Rules

We discuss law of large numbers, empirical probability and classical probability. We discuss statistical experiment, sample space and events. We define probability.

Exercise 5.1 (Probability Rules)
1. Terminology.
   (a) Summary.
   • Law of large numbers:
     Repeatedly sampling, sample average (proportion) will approach and stay close to expected population average (probability).
   • Empirical probability:
     Probability event (E) in experiment approximated by
     \[ P(E) \approx \frac{\text{frequency of } E}{\text{number of trials of experiment}}. \]
   • Classical probability:
     Probability event (E) in experiment, assuming equally likely outcomes,
     \[ P(E) = \frac{\text{number of ways } E \text{ occurs}}{\text{number outcomes in experiment}}. \]
   • Sample space, \( S \).
     List of all possible outcomes (or simple events) of experiment.
   • Event, \( E \); simple event, \( e_i \).
     Event is subset of all possible outcomes of experiment; simple event if subset consists of one outcome.
   • Probability experiment.
     Process which results in sample space where each outcome is assigned a chance of occurrence.
(b) Coin tossing.

![Graph showing proportion heads vs coin tosses](image)

<table>
<thead>
<tr>
<th>Coin tosses (flips)</th>
<th>Empirical coin frequency</th>
<th>Empirical probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5065</td>
<td>0.5065</td>
</tr>
<tr>
<td>100</td>
<td>4935</td>
<td>0.4935</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Classical coin outcomes</th>
<th>Classical probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>heads</td>
<td>1/2 = 0.5</td>
</tr>
<tr>
<td>tails</td>
<td>1/2 = 0.5</td>
</tr>
</tbody>
</table>

**Figure 5.1 (Terminology: coin tossing)**

i. True / False. As number of coin tosses increase, proportion of total tosses which are heads will approach and stay close to expected probability of tossing a head. This is an example of law of large numbers.

ii. Approximating probability with empirical approach\(^1\).

Since 5065 tosses of 10000 tosses are heads we approximate probability of tossing a head by \( P(H) \approx \frac{5065}{10000} = 0.4935 / 0.5 / 0.5065 \).

iii. Calculating probability with classical method\(^2\).

Since a coin can be tossed only as a head (H) or tail (T) and assuming equally likely outcomes, \( P(H) = \frac{1}{2} = 0.4935 / 0.5 / 0.5065 \).


Flipping a coin is a probability experiment. It is generally unknown, when flipping a coin, whether coin comes up heads (Hs) or tails (Ts). However, it is known there are only two possible outcomes (choose one) \{H, T\} / \{H, H\} / \{T, T\}.

v. Classical method: sample space\(^3\).

Sample space for flipping a coin is \( S = \{H, T\} \). Sample space is (choose one) set / subset / element of all possible outcomes.

vi. Classical method: event, simple event.

Flipping a head \{H\} is an example of an event, \( E \). An event is a (choose one) set / subset / element of all possible outcomes. Since only one outcome, \{H\}, this event is a simple event, \( E = e_1 = \{H\} \)

---

\(^1\)Accuracy of empirical method depends on how well we can consistently flip the coin.

\(^2\)Accuracy of classical method depends on accuracy of equally likely outcomes assumption.

\(^3\)Displayed as a Venn diagram in figure.
Section 1. Probability Rules (Lecture Notes 5)

Figure 5.2 (Terminology: die rolling)

i. **True / False.** As number of die rolls increase, proportion of total rolls which are 4s will approach and stay close to expected probability of rolling a 4. This is an example of law of large numbers.

ii. **Approximating probability with empirical approach.**
    Since 9900 tosses of 60000 rolls are 4s we approximate probability of rolling a 4 by $P(4) \approx \frac{9900}{60000} = (\text{choose one}) \ 0.163 / 0.164 / 0.165$.

iii. **Calculating probability with classical method.**
    Since a die can be rolled either 1, 2, 3, 4, 5 or 6 and assuming equally likely outcomes, $P(4) = \frac{1}{6} \approx (\text{choose one}) \ 0.163 / 0.165 / 0.167$.

iv. **Classical method: experiment.**
    Die rolling is probability experiment. Value to be rolled unknown, but six possible outcomes (choose one) known / unknown.

v. **Classical method: sample space.**
    Sample space is (choose one) $\{1, 2, 3, 4, 5\} / \{1, 2, 3, 4, 5, 6\}$.

vi. **Classical method: event, simple event.**
    Examples of events are (choose one or more!) $\{1\} / \{1, 2\} / \{1, 2, 3\}$.

(d) **Chance error, % chance error and law of large numbers: die rolling**
    Results of die rolling experiment given in table below.
### Chapter 5. Probability (Lecture Notes 5)

<table>
<thead>
<tr>
<th>no of rolls</th>
<th>no of 4s</th>
<th>expected 4s</th>
<th>difference</th>
<th>% difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>9</td>
<td>10</td>
<td>-1</td>
<td>-1.7%</td>
</tr>
<tr>
<td>120</td>
<td>21</td>
<td>20</td>
<td>1</td>
<td>0.8%</td>
</tr>
<tr>
<td>180</td>
<td>34</td>
<td>30</td>
<td>4</td>
<td>2.2%</td>
</tr>
<tr>
<td>240</td>
<td>42</td>
<td>40</td>
<td>2</td>
<td>0.8%</td>
</tr>
<tr>
<td>600</td>
<td>110</td>
<td>100</td>
<td>10</td>
<td>1.7%</td>
</tr>
<tr>
<td>1200</td>
<td>234</td>
<td>200</td>
<td>34</td>
<td>2.8%</td>
</tr>
<tr>
<td>2400</td>
<td>390</td>
<td>400</td>
<td>-10</td>
<td>-0.4%</td>
</tr>
</tbody>
</table>

i. After 60 rolls, $\frac{6}{7}$ are 4s; after 2400 rolls, $\frac{390}{400}$ are 4s.

ii. If fair, after 60 rolls, we expect $\frac{60}{6} = \frac{4}{5}$ are 4s; after 2400 rolls, we expect $\frac{2400}{6} = \frac{390}{400}$ are 4s.

iii. Chance error after 60 rolls $9 - 10 = -1$ are 4s; after 2400 rolls, chance error $390 - 400 = -10$ are 4s.

iv. Chance error **small / large / same** for a small number of rolls but **smaller / same / larger** for a large number of rolls.

v. Percentage chance error, 60 rolls $\frac{9-10}{60} \times 100 = -1.7\%$; after 2400 rolls, $\frac{390-400}{2400} \times 100 \approx -0.4\%$.

vi. Percentage of chance error **increases / decreases / remains same** as number of rolls increases: observed variability around $\frac{1}{6}$ decreases.

vii. So **observed** proportion of 4s converges towards / diverges from the **expected** proportion of 4s, $\frac{1}{6}$, by the law of large numbers.

viii. Law of large numbers **does / does not** describe the **expected variability** of chance error around $\frac{1}{6}$, but is known to be a bell-shaped histogram.

2. **Rules of probability.**

   (a) **Rules.**

   - Probability of any event, $E$, must be between 0 and 1: $0 \leq P(E) \leq 1$.
   - Sum of probability of all outcomes equals 1; for sample space $S = \{e_1, e_2, \ldots, e_n\}$,
     
     $$P(e_1) + P(e_2) + \cdots + P(e_n) = 1.$$

   (b) **Flipping coin three times**
Figure 5.3 (Probability: flipping coin three times)

i. Assuming each outcome equally likely,
\[ P(A) = P(HHH) = \text{(circle one)} \frac{1}{8} / \frac{3}{8} / \frac{4}{8} / \frac{5}{8} \]

ii. Probability flipping exactly one head,
\[ P(B) = P(HTT, THT, TTH) = \text{(circle one)} \frac{1}{8} / \frac{3}{8} / \frac{4}{8} / \frac{5}{8}. \]

iii. \( P(C) = \text{(circle one)} \frac{1}{8} / \frac{4}{8} / \frac{6}{8} / \frac{8}{8} \).

iv. Events A, B and C (choose one)
- overlap or intersect
- do not overlap, are disjoint (mutually exclusive)
- from one another.

v. Taken together, events A, B and C
- cover or shade
- do not cover or do not completely shade
- entire sample space.

vi. Rules of probabilities (choose one) violated / obeyed
since \( 0 \leq P(E) \leq 1 \), for \( E = A, B, C \) and \( P(A) + P(B) + P(C) = 1 \).

(c) Box of coins.

\[
\begin{array}{cccccccc}
\text{S} & \text{C} & \text{N} & \text{D} \\
\text{74c} & \text{78c} & \text{78c} & \text{76c} & \text{80c} \\
\text{74n} & \text{78n} & \text{80n} \\
\text{78d} & \text{81d} \\
\end{array}
\]

Figure 5.4 (Probability: choosing coins from a box)
i. Assuming each outcome equally likely, chance choosing a cent (penny) 
\[ P(C) = P(74c, 78c, 78c, 76c, 80c) = \text{(circle one) } \frac{2}{10} / \frac{3}{10} / \frac{5}{10}. \]

ii. Probability of choosing nickel, \( P(N) = \text{(circle one) } \frac{2}{10} / \frac{3}{10} / \frac{5}{10}. \)

iii. Probability of choosing dime, \( P(D) = \text{(circle one) } \frac{2}{10} / \frac{3}{10} / \frac{5}{10}. \)

iv. Rules of probabilities (choose one) violated / obeyed
since \( 0 \leq P(E) \leq 1, \) for \( E = C, N, D \) and \( P(C) + P(N) + P(D) = 1. \)

v. Probability choosing quarter is impossible since no quarters in box, so 
\( P(\text{quarter}) = \text{(choose one) } 0 / \frac{1}{10}. \)

vi. Probability choosing cent, nickel or dime is certain, so 
\( P(C \text{ or } N \text{ or } D) = \text{(choose one) } 0 / \frac{1}{10} / \frac{2}{10} / 1. \)

### 5.2 The Addition Rule and Complements

We discuss “and”, “or” and “not” (complement). We look at addition rule for disjoint events, general addition rule and complement rule for probability.

**Exercise 5.2 (The Addition Rule and Complements)**

1. **And, Or and Not (Complement).**

   (a) Summary of operations.
   - **And:** outcomes common to events; in intersection of events
   - **Or:** outcomes in union of events
   - **Not (Complement):** outcomes not in event, but in sample space
   - **Complement rule:** \( P(E^c) = 1 - P(E) \)

   (b) Box of tickets.
   Box has six tickets. Each ticket has 1, 2 or 3 with one of three subscripts: \( a, b \) or \( c. \) One ticket drawn from box at random.

   \[
   \begin{array}{cccccc}
   1_a & 2_a & 1_b & 3_b & 2_c & 3_c \\
   \end{array}
   \]

   Probability ticket is
   - i. “1” is \( P(1) = \text{ (circle one) } \frac{1}{6} / \frac{2}{6} / \frac{3}{6}. \)
   - ii. “a” is \( P(a) = \text{ (circle one) } \frac{1}{6} / \frac{2}{6} / \frac{3}{6}. \)
   - iii. “1” and an “a” is \( P(1 \text{ and } a) = \text{ (circle one) } \frac{1}{6} / \frac{2}{6} / \frac{3}{6}. \)
   - iv. “1” or an “a” is \( P(1 \text{ or } a) = \text{ (circle one) } \frac{1}{6} / \frac{2}{6} / \frac{3}{6}. \)
   - v. “1” and a “2” is \( P(1 \text{ and } 2) = \text{ (circle one) } \frac{0}{6} / \frac{1}{6} / \frac{2}{6}. \)
   - vi. not an “a” is \( P(a^c) = 1 - P(a) = 1 - \frac{2}{6} = \text{ (circle one) } \frac{2}{6} / \frac{3}{6} / \frac{4}{6}. \)
(c) Box of coins.
Coins are sampled at random from box.

\[
\begin{array}{cccccc}
74c & 78c & 78c & 76c & 80c \\
74n & 78n & 80n \\
78d & 81d \\
\end{array}
\]

Figure 5.5 (And, or, not: box of coins)

Chance coin is a

i. cent is \( P(C) = \) (circle one) \( \frac{3}{10} / \frac{4}{10} / \frac{5}{10} / \frac{6}{10} \).

ii. 1978 is \( P(1978) = \) (circle one) \( \frac{3}{10} / \frac{4}{10} / \frac{5}{10} / \frac{6}{10} \).

iii. cent and a 1978 is \( P(C \text{ and } 1978) = \) (circle one) \( \frac{1}{10} / \frac{2}{10} / \frac{3}{10} / \frac{4}{10} \).

iv. cent or a nickel is \( P(C \text{ or } N) = \) (circle one) \( \frac{5}{10} / \frac{6}{10} / \frac{7}{10} / \frac{8}{10} \).

v. cent or a 1978 is \( P(C \text{ or } 1978) = \) (circle one) \( \frac{5}{10} / \frac{6}{10} / \frac{7}{10} / \frac{8}{10} \).

vi. not a dime is \( P(D^c) = \) (circle one) \( \frac{5}{10} / \frac{6}{10} / \frac{7}{10} / \frac{8}{10} \).

2. Addition rule.

(a) Summary.

- Addition rule for disjoint events \( E, F \).

\[
P(E \text{ or } F) = P(E) + P(F)
\]

- General addition rule for events \( E, F \).

\[
P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)
\]

(b) Box of coins.
Refer to figure above.

i. Since not possible to choose a single coin that is both a cent and a nickel; in other words, choosing a cent and nickel are mutually exclusive (disjoint) events, probability of choosing cent or a nickel is

\[
P(C \text{ or } N) = P(C) + P(N) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10}
\]

(circle one) \( \frac{5}{10} / \frac{6}{10} / \frac{7}{10} / \frac{8}{10} \).
ii. Since it is possible to choose a single coin that is both a cent and a 1978; in other words, choosing a cent and 1978 coin is not disjoint, probability of choosing a cent or a 1978 is

\[ P(C \text{ or } 1978) = P(C) + P(1978) - P(C \text{ and } 1978) = \frac{5}{10} + \frac{4}{10} - \frac{2}{10} = \text{(circle one) } \frac{5}{10} / \frac{6}{10} / \frac{7}{10} / \frac{8}{10}. \]

iii. \( P(C \text{ or } D) = P(C) + P(D) = \frac{5}{10} + \frac{2}{10} = \text{(circle one) } \frac{5}{10} / \frac{6}{10} / \frac{7}{10} / \frac{8}{10}. \)

iv. \( P(1978 \text{ or } D) = P(1978) + P(D) - P(1978 \text{ and } D) = \frac{4}{10} + \frac{2}{10} - \frac{1}{10} = \text{(circle one) } \frac{5}{10} / \frac{6}{10} / \frac{7}{10} / \frac{8}{10}. \)

(c) Addition rule: fathers, sons and college.

Data from a sample of 80 families in a midwestern city gives record of college attendance by fathers (F) and their oldest sons (S).

<table>
<thead>
<tr>
<th>father attended college</th>
<th>son attended college</th>
<th>son did not attend college</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>22</td>
<td>33</td>
<td>55</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>80</td>
</tr>
</tbody>
</table>

i. Probability son, in a randomly chosen family, attended college, is

\[ P(S) = \text{(circle one) } \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{25}{80}. \]

ii. Probability father, in a randomly chosen family, attended college, is

\[ P(F) = \text{(circle one) } \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{25}{80}. \]

iii. Probability son and father both attended college is

\[ P(S \text{ and } F) = \text{(circle one) } \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{18}{80}. \]

iv. Probability son or father both attended college is

\[ P(S \text{ or } F) = P(S) + P(F) - P(S \text{ and } F) = \frac{40}{80} + \frac{25}{80} - \frac{18}{80} = \text{(choose one) } \frac{45}{80} / \frac{46}{80} / \frac{47}{80} / \frac{48}{80}. \]

(d) More addition rule.

i. If \( P(E) = \frac{3}{36} \), \( P(F) = \frac{9}{36} \) and \( P(E \text{ and } F) = \frac{2}{36} \),

\[ P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) = \frac{10}{36} / \frac{11}{36} / \frac{12}{36}. \]

ii. \( P(E) = 0.25 \), \( P(F) = 0.10 \) and \( P(E \text{ and } F) = 0.03 \),

\[ P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) = 0.30 / 0.32 / 0.33. \]

iii. True / False. Addition rule determines chance of E “or” F.

iv. True / False. Events E and F are disjoint if \( P(E \text{ and } F) = 0 \).

5.3 Independence and the Multiplication Rule

We look at probability of independent events (sampling with replacement) and associated (special case of) multiplication rule: if \( E, F, G \) are independent, then

\[ P(E \text{ and } F \text{ and } G \text{ and } \ldots) = P(E) \cdot P(F) \cdot P(G) \ldots \]
Exercise 5.3 (Independence and the Multiplication Rule)

1. Independence (sampling with replacement)  
   versus dependence (sampling without replacement): box of tickets.  
   Two things are independent if chance for second given first are the same, no  
   matter how first turns out; otherwise, two things are dependent.

   \[
   \begin{array}{cccc}
   a & b & c & 2c \\
   1a & 2a & 1b & 3b & 2c & 3c
   \end{array}
   \]

   (a) Sample with replacement: independence.  
   Two tickets are sampled with replacement at random from box. All six  
   tickets remain in box when second ticket is drawn. Chance second ticket  
   is a “2” given first ticket is a “1” is (circle one) \( \frac{1}{6} / \frac{2}{6} / \frac{3}{6} \).

   (b) When sampling at random with replacement, chance second ticket of two  
   drawn from box is “2”, no matter what the first, is always \( \frac{1}{6} / \frac{2}{6} / \frac{3}{6} \).

   (c) Sample without replacement: dependence.  
   Two tickets are sampled without replacement at random from box. Only five  
   tickets remain in box when second ticket is drawn. Chance second ticket is a  
   “2” given first ticket is a “1” is \( \frac{1}{5} / \frac{2}{5} / \frac{3}{5} \).

   (d) True / False When sampling at random without replacement, chance  
   second ticket of two drawn from box is any given number depends on  
   number drawn on first ticket.

   (e) When sampling at random without replacement, draws are  
   (circle one) independent / dependent of one another;  
   with replacement, draws are independent.

   (f) True / False. If small random samples without replacement are taken  
   from large populations (sample size less than 5% of population), it is rea-  
   sonable to assume independence of events because fractions change so little.


   \[
   \begin{array}{cccc}
   a & b & c & 2c \\
   1a & 2a & 1b & 3b & 2c & 3c
   \end{array}
   \]

   (a) Two tickets are sampled with replacement from box: tickets are independent  
   of one another. Chance first ticket is “1” and second ticket is “3” is

   \[
P(1 \text{ and } 3) = P(1) \cdot P(3) = (\text{circle one}) \frac{1}{6} \times \frac{1}{6} / \frac{2}{6} \times \frac{2}{5} / \frac{2}{6} \times \frac{2}{5} = \frac{1}{5}.
\]

   This is an example of (a special case of) the multiplication rule.

---

4 Order of tickets matters here. If you did not see how two tickets were picked from box, it would  
not be clear whether “1” or “3” was first or second ticket and so, in this case,  
\[P(1 \text{ and } 3) + P(3 \text{ and } 1) = P(1) \cdot P(3) + P(3) \cdot P(1) = 2 \times \frac{2}{6} \times \frac{2}{6}.\]
(b) Two tickets are sampled without replacement from box: tickets are dependent of one another. Chance first ticket is “1” and second ticket is “3” is
\[ P(1 \text{ and } 3) = P(1) \cdot P(3|1) = \text{(circle one)} \frac{1}{6} \times \frac{1}{6} / \frac{2}{6} \times \frac{2}{6} / \frac{2}{6} \times \frac{2}{6} = \frac{1}{15}. \]
This is an example of the (general) multiplication rule.

(c) True / False
If “1” and “3” independent, then \( P(1 \text{ and } 3) = P(1) \cdot P(3) \).
If “1” and “3” dependent, then \( P(1 \text{ and } 3) = P(1) \cdot P(3|1) \neq P(1) \cdot P(3) \).
In fact, the reverse is true: If \( P(1 \text{ and } 3) = P(1) \cdot P(3) \),
then “1” and “3” independent, otherwise they are dependent.

(d) Three tickets are sampled with replacement from box. Chance first ticket is “1” and second ticket is “3” and third ticket is “3” is
\[ P(1 \text{ and } 3 \text{ and } 3) = P(1) \cdot P(3) \cdot P(3) = \frac{2}{6} / \frac{2}{6} \times \frac{2}{6} / \frac{2}{6} \times \frac{2}{6} = \frac{1}{27}. \]

(e) Three tickets are sampled with replacement from box. Chance all three tickets are “3”s is (circle one) \( \frac{2}{6} / \frac{2}{6} \times \frac{2}{6} / \frac{2}{6} \times \frac{2}{6} = \frac{1}{27}. \)

(f) Three tickets are sampled with replacement from box. Chance at least one of the three tickets is a “3” is either
- chance one ticket of three is a “3” or two tickets of three are “3”s or
three tickets of three are “3”s, OR
- one minus chance none of three tickets are “3”s,
\[ 1 - P(\text{no 3s}) = (\text{circle one}) 1 - \frac{4}{6} / 1 - \frac{4}{6} \times \frac{4}{6} / 1 - \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} = \frac{19}{27}. \]

3. Independence versus dependence: fathers, sons and college.

<table>
<thead>
<tr>
<th></th>
<th>son attended college</th>
<th>son did not attend college</th>
</tr>
</thead>
<tbody>
<tr>
<td>father attended college</td>
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<tr>
<td></td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>

(a) Probability son, in a randomly chosen family, attended college, is
\[ P(S) = (\text{circle one}) \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{25}{80}. \]

(b) Probability father, in a randomly chosen family, attended college, is
\[ P(F) = (\text{circle one}) \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{25}{80}. \]

(c) Probability son and father both attended college is
\[ P(S \text{ and } F) = (\text{circle one}) \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{18}{80}. \]

\(^5\)Order of the tickets does not matter here, whether you saw in what order the tickets were chosen or not, since all three tickets are the same: all “3”s. The answer will always be \( \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} \) in this case.
Section 4. Conditional Probability and the General Multiplication Rule (Lecture Notes 5)

(d) Since \( \frac{18}{80} \neq \left( \frac{40}{80} \right) \times \left( \frac{25}{80} \right) \) or 0.225 \( \neq 0.15625 \); in other words, 
\[ P(S \text{ and } F) \neq P(S) \times P(F), \]
the event “son attends college”
(circle one) is **independent of** / **depends on** event “father attends college”.

(e) **True** / **False** If \( P(A \text{ and } B) = P(A) \times P(B) \),
then events \( A \) and \( B \) are independent, otherwise they are dependent.
This is one method to determine independence/dependence.

4. **Independence versus disjoint events.** **True** / **False**.
Independent events are different from disjoint events.
Events \( E \) and \( F \) are independent if \( P(E \text{ and } F) = P(E) \times P(F) \).
Events \( E \) and \( F \) are disjoint if \( P(E \text{ and } F) = 0 \).

5.4 Conditional Probability and the General Multiplication Rule

We will look at conditional probability

\[
P(F|E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{P(F \text{ and } E)}{P(E)},
\]
and general multiplication rule,

\[
P(E \text{ and } F) = P(E) \times P(F|E).
\]

Exercise 5.4(Conditional Probability and the General Multiplication Rule)

1. **Conditional probability and dependence: box of coins.**

![Box of coins](image)

Figure 5.6 (Conditional probability: box of coins)

(a) **Choosing 1974.**

Chance a coin chosen at random from box is a 1974 coin is
\[ P(1974) = (\text{circle one}) \frac{1}{10} / \frac{2}{10} / \frac{3}{10} / \frac{4}{10}. \]
(b) **Choosing 1974, given nickel.**
Of three coins that are nickels, (circle one) \( \frac{1}{3} \) / \( \frac{2}{3} \) / \( \frac{3}{3} \) are 1974 coins. Given coin taken from box is a nickel, chance this coin is a 1974 nickel is \( P(1974|N) = (\text{circle one}) \frac{1}{3} / \frac{2}{3} / \frac{3}{3} \).

(c) **Choosing 1974 depends on choosing nickel.**
Unconditional chance coin is “1974”, \( P(1974) = \frac{2}{10} \), is (circle one) equal / not equal to conditional chance coin is “1974, given a nickel”, \( P(1974|N) = \frac{1}{3} \). Choosing a “1974” and choosing a ”nickel” are dependent.

(d) **Choosing cent.**
Chance of choosing a cent is \( P(C) = (\text{circle one}) \frac{2}{5} / \frac{4}{10} / \frac{8}{20} \).

(e) **Choosing cent, given 1978.**
Of coins that are 1978s, (circle one) \( \frac{2}{5} / \frac{4}{10} / \frac{8}{20} \) are cent coins. Given a coin is a 1978, chance this coin is a cent is \( P(C|1978) = \frac{2}{5} / \frac{4}{10} / \frac{8}{20} \).

(f) **Choosing cent independent of choosing 1978.**
Since \( P(C) = \frac{5}{10} = P(C|1978) = \frac{2}{4} \), choosing a “cent” and choosing a ”1978” are (choose one) independent / dependent events.

(g) **In general.**
If \( P(E) = P(E|F) \), events \( E \) and \( F \) are dependent / independent; otherwise, they are dependent. This is a second method to determine independence/dependence.

2. **More conditional chance: fathers, sons and college.**

<table>
<thead>
<tr>
<th></th>
<th>son attended college, ( S )</th>
<th>son did not attend college, ( S^c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>father attended college, ( F )</td>
<td>18</td>
<td>7</td>
</tr>
<tr>
<td>father did not attend college, ( F^c )</td>
<td>22</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

(a) Probability a son attended college given a father attended college, in a randomly chosen family, is \( P(S \mid F) = (\text{circle one}) \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{25}{80} \).

(b) Probability a son did not attend college given a father attended college, in a randomly chosen family, is \( P(S^c \mid F) = (\text{circle one}) \frac{7}{25} / \frac{18}{25} / \frac{7}{18} / \frac{25}{7} \).

(c) \( P(S \mid F^c) = (\text{circle one}) \frac{55}{22} / \frac{33}{55} / \frac{22}{55} / \frac{22}{80} \).

(d) \( P(S^c \mid F^c) = (\text{circle one}) \frac{22}{55} / \frac{33}{80} / \frac{22}{33} / \frac{33}{55} \).

(e) \( P(F \mid S) = (\text{circle one}) \frac{18}{40} / \frac{18}{25} / \frac{18}{22} / \frac{25}{80} \) (circle one) equals / does not equal \( P(S \mid F) = \frac{18}{25} \).
(f) Using the formula.

\[ P(S \mid F) = \frac{P(S \text{ and } F)}{P(F)} = \frac{18/80}{25/80} = \]

(circle one) \( \frac{18}{40} \) / \( \frac{18}{25} \) / \( \frac{40}{80} \) / \( \frac{25}{80} \).

3. General multiplication rule. A deck is shuffled and three cards are dealt.

(a) Chance first card dealt is an ace is

\[ P(\text{ace}) = (\text{circle one}) \frac{1}{52} / \frac{4}{52} / \frac{3}{51} / \frac{1}{51}. \]

(b) Chance second card dealt is a jack, given first card dealt is an ace, is

\[ P(\text{jack} \mid \text{ace}) = (\text{circle one}) \frac{1}{52} / \frac{4}{50} / \frac{3}{51} / \frac{1}{51}. \] This is conditional probability since chance of one event depends on occurrence of another event.

(c) Probability first card is an ace and second card is a jack is

\[ P(\text{ace and jack}) = P(\text{ace}) \cdot P(\text{jack} \mid \text{ace}) = \]

(circle one) \( \frac{1}{52} \times \frac{3}{51} / \frac{4}{52} \times \frac{4}{51} / \frac{4}{51} / \frac{1}{51}. \) This is an example of general multiplication rule because it involves product of unconditional probability and conditional probability.

(d) Probability third card dealt is a jack, conditional on first two cards dealt are a jack and an ace, is

\[ P(\text{jack} \mid (\text{ace and jack})) = (\text{circle one}) \frac{1}{50} / \frac{4}{52} / \frac{3}{51} / \frac{3}{50}. \] This is another example of a conditional probability.

(e) Probability of an ace, jack and another jack is

\[ P(\text{ace and jack and jack}) = P(\text{ace}) \cdot P(\text{jack} \mid \text{ace}) \cdot P(\text{jack} \mid (\text{ace and jack})) = \]

\[ \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}, \frac{4}{52} \times \frac{4}{51} \times \frac{3}{50}, \frac{4}{50} \times \frac{3}{49} \times \frac{2}{48}. \] This is another example of the general multiplication rule\(^6\).


When “and” is involved, “multiply": \( P(E \text{ and } F) = P(E) \cdot P(F \mid E) \).

When “or” is involved, “add”: \( P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) \);

5.5 Counting Techniques

Counting can be an important part of determining various probabilities. To help in understanding various counting techniques, we first discuss the notion of whether

\(^6\)The order of the cards matters here. If you did not see how the three cards were picked from the deck, it would not be clear which one of three possibilities occurred: ace, jack, jack or jack, ace, jack or jack, jack, ace. In this case, the answer would be \( 3 \times \frac{4}{52} \times \frac{4}{51} \times \frac{3}{50} \).
order matters or not. Then, we look at two counting rules: permutations and combinations. Along the way, we find out two ways of visualizing counting techniques: tree diagrams and “marbles in a box”.

**Exercise 5.5 (Counting Techniques)**

1. **Summary.**
   - **Multiplication rule of counting:** If \( n_1 \) possible first choices, \( n_2 \) second choices and so on, then total of \( n_1 \cdot n_2 \cdot ... \) choices.
   - **Permutation:** count of ordered arrangement of \( r \) from \( n \) distinct objects, sampled without replacement
     \[
     nP_r = \frac{n!}{(n-r)!}
     \]
   - **Combination:** count of unordered arrangement of \( r \) from \( n \) distinct objects, sampled without replacement
     \[
     nC_r = \frac{n!}{r!(n-r)!}
     \]
   - **Permutation of indistinct items:** count of ordered arrangement of \( n \) distinct objects into \( n_1, n_2, \ldots, n_k \) groups, sampled without replacement
     \[
     \frac{n!}{n_1!n_2!\cdots n_k!}
     \]
   - **Permutation of distinct items with replacement:** count of ordered arrangement of \( r \) of \( n \) distinct objects, sampled with replacement
     \[
     n^r
     \]

2. **Order matters or not when counting.**
   
   (a) **Committees (Generic).** Three possible committees of two people from Jim, Sue and Ali are (Jim, Sue), (Jim, Ali) and (Sue, Ali). We (choose one) **would / would not** include both (Jim, Sue) and (Sue, Jim) in the count because **order** of people chosen for this committee **does not matter**.

   (b) **Committees (Different Roles).** We (choose one) **would / would not** include (Jim, Sue) and (Sue, Jim) in a count of (chair, secretary) committees because of two roles in committee; **order** of people chosen **does matter**.

   (c) **Street numbers.** When counting 3–digit street numbers, (circle one) **order matters / order does not matter**. Are (9,3,4) and (3,4,9) two street numbers (order matters) or one street number (order does not matter)?
(d) Cards. When dealing five cards for a poker hand, (circle one) order matters / order does not matter. Are (10,J,Q,2,3) and (2,3,J,Q,10) two hands (order matters) or one hand (order does not matter)?

(e) Cars. When parking cars in parking spots (P1, P2, P3) where P1, P2 and P3 have different parking fees, (circle one) order matters / order does not matter. Is parking (Ford,GM,Toyota) same arrangement as parking (GM,Toyota,Ford) in (P1, P2, P3)?

(f) Multiple-choice quiz questions. When answering a sequence of multiple-choice questions on a quiz, (circle one) order matters / order does not matter. One possible sequence of answers on a quiz with five questions is A, A, D, A, C and another is D, A, A, C, A, for example.

(g) We count more / less if we assume order matters.

3. Marbles in a box used for counting outcomes.

(a) Number of 3-digit street numbers, as indicated in “marbles in boxes” counting method below: (choose two!) \(10 \times 10 \times 10 / 100 / 1000\).

(b) Number of three-digit numbers if first number cannot be zero: (circle one or more!) \(9 \times 10 \times 10 / 900 / 999\).

(c) Number of three-digit numbers if first number must be 3: (circle one or more!) \(1 \times 10 \times 10 / 100 / 900\).

(d) Number three-digit numbers if first number is 3, second cannot be 9: (circle one) \(9 \times 9 \times 10 / 9 \times 10 \times 10 / 1 \times 9 \times 10\).

(e) Number of four-digit numbers if first number cannot be zero: (circle one) \(9 \times 10 \times 10 \times 10 \times 1 \times 9 \times 10 \times 10\).

(f) Number of three-digit numbers with exactly two 3s: (circle one) \(1 \times 1 \times 9 = 9 / 1 \times 9 \times 1 = 9 / 9 \times 1 \times 1 = 9 / 9 + 9 + 9 = 27\).

Hint: Correct answer is fourth one, 27, and is obtained by adding first three possible answers together. Each of first three answers represent a different way to have exactly two 3s in three digits.
4. **Tree diagrams used for multiplication rule of counting outcomes.**

(a) As shown in tree diagram, there are (choose one) $3 + 2 = 5$ / $3 \times 2 = 6$
possible (treasurer, secretary) pairs from three eligible treasurer candidates $(T_1, T_2, T_3)$ and two eligible secretary candidates $(S_1, S_2)$, including
{$(T_1, S_1)$, $(T_1, S_2)$, $(T_2, S_1)$, $(T_2, S_2)$, $(T_3, S_1)$, $(T_3, S_2)$}.

(b) Number of treasurer and secretary pairs, when *four* eligible treasurer candidates, *eight* eligible secretary candidates, is (circle one) $4 + 8 = 12$ / $4 \times 8 = 24$ / $24$.

(c) Number of treasurer, secretary and president triplets, when *three* eligible treasurer candidates, *five* eligible secretary candidates and *two* eligible president candidates (circle one) $3 + 5 + 2 = 10$ / $20$ / $3 \times 5 \times 2 = 30$.

5. **Factorial notation used in formulas for counting outcomes.**

(a) Special mathematical notation, called *factorial notation*, denoted by an
exclamation mark, “!” , is often used in formulas used to count outcomes
in a sample space. For example,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 =$$

(choose one) $100$ / $110$ / $120$.

(StatCrunch: Data, Compute expression, fact(5), Compute.)

(b) $7!$ is equal to (circle none, one or more)

i. $7 \times 6!$

ii. $5040$

iii. $7 \times 6 \times 5!$

iv. $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

(StatCrunch: Data, Compute expression, fact(7), Compute.)
(c) \( \frac{7!}{5!} \) is equal to (circle none, one or more)
   i. \( \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \)
   ii. \( 7 \times 6 \)
   iii. 42

(d) \( \frac{7!}{5!3!} \) is equal to (circle none, one or more)
   i. \( \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(5 \times 4 \times 3 \times 2 \times 1)(3 \times 2 \times 1)} \)
   ii. \( \frac{7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} \)
   iii. \( \frac{7 \times 6}{3 \times 2 \times 1} \)
   iv. \( \frac{42}{6} \)

(e) \( (7 - 3)! \) is equal to (circle none, one or more) \( 7! - 3! / 4! / 4 \times 3 \times 2 \times 1 / 24 \).

(f) \( \frac{7!}{(7 - 3)!} \) is equal to (circle none, one or more)
   i. \( \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \)
   ii. \( 7 \times 6 \times 5 \)
   iii. 210

(g) By definition (in other words, accept as true that), 0! = 1, and so 0! is equal to (circle one) \( 1! / 2! / 3! \).

6. Permutation formula (order matters) used in counting outcomes.

(a) How many different ways can three of five cars be parked in three different side-by-side parking spots with three different parking fees? As shown in marbles in boxes diagram below, since five different cars occupy first parking spot, only four occupy second parking spot (since one car is in first parking spot) and three could occupy final parking spot, number of permutations is (circle one) \( 5 + 4 + 3 = 12 / 5 \times 4 = 20 / 5 \times 4 \times 3 = 60 \).

![Diagram of marbles and parking spots](image-url)
(b) **True / False** In marbles in boxes analogy, side–by–side boxes represent parking spots and marbles represent cars, A, B, C, D and E.

(c) Since (A,B,C) and (B,A,C) count as two different parking arrangements, even though same three cars are used for both, “marbles in boxes” counting method assumes (circle one) order matters / order does not matter.

(d) Number of ways to park 5 cars in 4 parking spots (circle one)
\[ 5 \times 4 \times 3 \times 2 = 120 \] / \[ 4 \times 3 \times 2 \times 1 = 24 \] / \[ 5 \times 4 \times 3 = 120 \].

(e) Number of ways to park \( n = 5 \) cars in \( r = 4 \) spots (circle one or more)
   - i. \( \frac{5 \times 4 \times 3 \times 2 \times 1}{1} \)
   - ii. \( \frac{5!}{1!} \)
   - iii. \( nP_r = \frac{n!}{(n-r)!} \), where \( n = 5 \) and \( r = 4 \)

(f) Number of ways to park \( n = 7 \) cars in \( r = 4 \) spots (circle one or more)
   - i. \( nP_r = \frac{n!}{(n-r)!} \), where \( n = 7 \) and \( r = 4 \)
   - ii. \( \frac{7!}{(7-4)!} \)
   - iii. \( \frac{7!}{3!} \)
   - iv. \( 7 \times 6 \times 5 \times 4 = 840 \)

(StatCrunch: Data, Compute expression, perm(7,4), Compute.)

(g) Number of ways to park \( n = 12 \) cars in \( r = 8 \) spots (circle one or more)
   - i. \( nP_r = \frac{n!}{(n-r)!} \), where \( n = 12 \) and \( r = 8 \)
   - ii. \( 12P_8 = \frac{12!}{(12-8)!} \)
   - iii. \( \frac{12!}{4!} \)
   - iv. \( 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 19,958,400 \)

7. **Combination formula (order does not matters) used in counting outcomes.**

(a) Number of ways of dealing three cards from five cards (10,J,Q,K,A) is calculated by assuming order matters (5 marbles, three side–by–side boxes),
\[ 5 \times 4 \times 3 = 60 \]
and then “dividing out the order” \( (3! = 6 \) permutations of three cards),
\[ \frac{5 \times 4 \times 3}{3!} = \frac{60}{6} = \]
(circle one) \( 9 / 10 / 11 \) combinations. See figure below.
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**Figure 5.10 (Combinations: number of hands in 5 cards)**

(b) Number of ways of dealing \( r = 3 \) of \( n = 5 \) cards (choose one or more)

i. \( \frac{5 \times 4 \times 3}{3!} \)

ii. \( \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1)3!} \)

iii. \( \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1)(3 \times 2 \times 1)} \)

iv. \( 5C_3 = \frac{5!}{2!3!} = 10 \)

v. \( nC_r = \frac{n!}{(n-r)!r!} \), where \( n = 5 \) and \( r = 3 \)

(c) Number of ways of dealing \( r = 3 \) of \( n = 11 \) cards (choose one or more)

i. \( nC_r = \frac{n!}{(n-r)!r!} \), where \( n = 11 \) and \( r = 3 \)

ii. \( \frac{11!}{(11-3)!3!} \)

iii. \( \frac{11!}{8!3!} \)

iv. \( \frac{11 \times 10 \times 9}{3 \times 2 \times 1} \)

v. \( 11C_3 = 165 \)

(a) As shown in figure below, chance four–digit PIN number drawn from a barrel begins with 3 is

\[
\frac{1 \times 10 \times 10 \times 10}{10 \times 10 \times 10 \times 10} = \frac{1000}{10000} = 0.1
\]

(circle one or more) \(\frac{1000}{10000} / 0.1 / 10\%\).

(b) Chance four–digit PIN number drawn from a barrel has exactly three 3s is

\[P(3 \text{ threes}) = \frac{36}{10000} / \frac{2916}{10000} / \frac{1}{10000} = 0.0036 \text{ or } 0.36\%\]

(look at last part of question 2 above.)

(c) Chance four–digit PIN number drawn from a barrel has exactly one 3 is

\[P(\text{one three}) = \frac{36}{10000} / \frac{2916}{10000} / \frac{1}{10000} = 0.2916 \text{ or } 29.16\%\]

(again, look at last part of question 2 above.)

(d) Chance four–digit PIN number drawn from a barrel is the number 1234 is

\[P(1234) = \frac{36}{10000} / \frac{2916}{10000} / \frac{1}{10000} = 0.0001 \text{ or } 0.01\%\]

This is unusual because \(P(1234) = 0.0001\) is small, smaller than 0.05.

9. Permutations in probability. Since there are \(5 \times 4 \times 3 = 60\) permutations for 3 of five cars, A, B, C, D and E, to park in 3 parking spots, and, furthermore, \(1 \times 4 \times 3 = 12\) ways for car A to park in the first spot (and two other cars to park in the other two spots), the chance car A parks in the first spot is \(\frac{12}{60} = 0.2\) or 20%. With this in mind, the chance that car A parks in the first spot, if ...

(a) ... 4 of 5 cars park in 4 parking spots, is

\[\text{(circle one) } \frac{1 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2} / \frac{5 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2} / \frac{1 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2} = \frac{1 \times 4 \times P_3}{5 \times P_4} = 0.20 \text{ or } 20\%\]
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Another combination and probability question: choosing people.

10. Combinations in probability. Since there are \( \binom{5}{3} = \frac{5!}{3!2!} = 10 \) combinations for dealing 3-card hands from five cards, 10, J, Q, K and A, and, furthermore, \( 1 \times 4 \binom{2}{1} = 6 \) ways for a jack to appear in a 3-card hand, chance a jack appears in a 3-card hand is \( \frac{1 \times 4 \binom{2}{1}}{\binom{5}{3}} \approx 0.1429 \) or 14.29%.

| (a) | Order (circle one) \textbf{matters} / \textbf{does not matter}. |
| (b) | Total number of ways 5 people can be chosen from 12 is \( \frac{12!}{5!7!} \). |
| (c) | Chance Scout, Buzz and Mary and two others are chosen from 12 is \( \frac{12!}{5!7!} \). |

10. Combinations in probability. Since there are \( \binom{5}{3} = \frac{5!}{3!2!} = 10 \) combinations for dealing 3-card hands from five cards, 10, J, Q, K and A, and, furthermore, \( 1 \times 4 \binom{2}{1} = 6 \) ways for a jack to appear in a 3-card hand, chance a jack appears in a 3-card hand is \( \frac{1 \times 4 \binom{2}{1}}{\binom{5}{3}} \approx 0.1429 \) or 14.29%.

11. Another combination and probability question: choosing people. Determine chance Scout, Buzz and Mary and two others are chosen from 12 people.

(a) Order (circle one) \textbf{matters} / \textbf{does not matter}.

(b) Total number of ways 5 people can be chosen from 12 is \( \frac{12!}{5!7!} \). 

(c) Chance Scout, Buzz and Mary and two others are chosen from 12 is \( \frac{12!}{5!7!} \). 

This is unusual because 4.55% is small, smaller than 5%.

12. Another probability question using counting: multiple choice quiz. On first question on a multiple choice quiz, of four possible choices (A, B, C and D), answer C might be correct, say. One possible answer key to the six questions on this quiz might be C, B, A, D, A and C, say. If an answer key is created at random, determine chance first two answers to this answer key are both A.

(a) Order (circle one) \textbf{matters} / \textbf{does not matter}.

(b) “Marbles” are (circle one) \textbf{six questions} / \textbf{four answers} and “boxes” are (circle one) \textbf{six questions} / \textbf{four answers}.

(c) If marbles are four answers and boxes are six questions, total number of possible different answer keys is (circle one) \( \frac{4^6}{6^4} \).

(d) Chance first two answers to this answer key are both A is (circle one) \( \frac{4^6}{6^4} \).
5.6 Putting It Together: Which Method Do I Use?

We will skip this section because we have covered everything in it. This section summarizes all material in previous sections.