Course Review Information
Mathematics 223

This course covers chapters 1 to 7 and 13, except 1.3, 6.3-6.6 and 7.4-7.6 in both my lecture notes and the text. There are four main parts to this course. What they are and where they are located in both the lecture notes and text are given below.

- *Linear and Nonlinear Equations* accounts for about 30% of course and is found in chapters 1 and 2 (lecture notes 1, 2, 3 and 4).
- *Derivatives* accounts for about 35% of the course and is found in chapter 3, 4 and 13 (lecture notes 5, 6, 7, 8 and 9).
- *Identifying Absolute and Relative Extrema* accounts for about 20% of the course and is found in chapters 5 and 6 (lecture notes 10, 11 and 12).
- *Integration* accounts for about 15% of the course and is found in chapters 7 and 13 (lecture notes 13 and 14).

Chapter 1. Linear Functions (lecture notes 1,2)

After discussing points and the *Cartesian coordinate system*, *linear functions* (lines) are discussed. In particular, the *slope* of a line is,

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

Linear functions can take different forms:

- slope-intercept: $y = mx + b$, slope m, y-intercept b
- point-slope form: $y - y_1 = m(x - x_1)$, slope m, line passes through (x_1, y_1)
- general form: $ax + by = c$, a, b, c integers with no common factor, $x \geq 0$
- vertical line: $x = k$, x-intercept k, undefined slope
- horizontal line: $y = k$, y-intercept k, zero slope

Lines are *parallel* if and only if slopes equal or all are vertical. Two lines are *perpendicular* if and only if product of slopes are -1, $m_1 \cdot m_2 = -1$ (or $m_2 = -\frac{1}{m_1}$) or one is vertical and the other is horizontal. Many real-world situations can be modeled by linear functions.

We consider linear *functions* in this section, such as $f(x) = 2x+4$, where $y = f(x)$ is the *dependent* variable and x is the *independent* variable. All linear *equations* are linear *functions* except equations of the form $x = k$ where k is a constant (Since $x = k$ is vertical, the slope is undefined and so this equation cannot also be a function). We look at how to solve two linear functions, to find their intersection, and, furthermore, give applications of solving linear functions. In particular, we look at economic supply, demand and equilibrium examples and also business cost and break-even analyses, as well as a finite mathematics feasible regions example. We also notice there are only three possible ways two lines can intersect: at one point, no point (inconsistent solution) or infinite points (dependent, identity solution).

Chapter 2. Nonlinear Functions (lecture notes 3,4)

A *function* is a rule which assigns to each element in one set one and only element from another set. A *non* linear function is *not* a straight line. A function be described in different ways, such as using a set diagram, table or by equation. The *domain* is the set of all possible values of the independent variable of a function x; the *range* is the set of all possible values of the dependent variable of a function $y = f(x)$. Two special types of functions are discussed, including:

- *even function*: $f(-x) = f(x)$, a function symmetric about the y-axis,
- *odd function*: $f(-x) = -f(x)$, a function symmetric about the origin.

Step functions are also discussed.

A *quadratic* function is defined as:

$$f(x) = ax^2 + bx + c$$

where a, b, c are real, $a \neq 0$. The graph of a quadratic is always a *parabola*. The maximum/minimum point (*vertex*) of quadratic/parabola is given at:

$$(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

Increasing c moves (*translates*) parabola upwards; decreasing c translates parabola downwards. *Negative* a flips (*reflects*) parabola downwards; *positive* a reflects parabola upwards. Increasing *magnitude* of a increases steepness of parabola. *Completing the square* of the quadratic, by factoring first two terms of quadratic then adding the square of one-half of the coefficient of x in the parentheses and subtracting outside, gives

$$y = a(x-h)^2 + k$$

where (h, k) is, again, the vertex of the parabola. For *any* function f and positive h and k,

- $y = f(x) + k$ is graph of $f(x)$ translated *up*wards by k
- $y = f(x) - k$ is graph of $f(x)$ translated *down*wards by k
- $y = f(x - h)$ is graph of $f(x)$ translated *right* (not left!) by h
- $y = f(x + h)$ is graph of $f(x)$ translated *left* by h
- $y = -f(x)$ is graph of $f(x)$ reflected upward down, vertically, across x-axis
- $y = f(-x)$ is graph of $f(x)$ reflected horizontally, across y-axis

A *polynomial* function of *degree* n is defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where leading coefficient $a_n \neq 0$, the a_i are real numbers and n is a nonnegative integer. Linear and quadratic functions are polynomials of degree 1 and 2, respectively; *cubic* and *quartic* polynomials are of degrees 3 and 4, respectively. Simple polynomials of the form $f(x) = x^n$ are called *power* functions. Some properties of polynomials:

- polynomials of degree n have *at most* $n - 1$ *turning points* (or *relative extrema*); graphs of polynomials with n turning points are *at least* of degree $n + 1$
- ends of a polynomial with *even* degree either both turn up or both turn down; one end of a polynomial with *odd* degree turns up and other turns down
- graph goes up as x becomes a large positive number if leading coefficient positive; goes down if leading coefficient negative

A *rational* function is

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Since

- if a function grows larger in magnitude as x approaches k, $x = k$ is a *vertical asymptote*,
- if a function approaches k as $|x|$ gets larger, $y = k$ is a *horizontal asymptote*,

then if both numerator $p(x)$ and denominator $q(x)$ of a rational function are zero at same $x = k$, graph has a hole (*removable discontinuity*) at k, but if only denominator $q(x)$ is zero at $x = k$, $x = k$ is a vertical asymptote.

An exponential function is given by

$$f(x) = a^x$$

where x is any real number, $a > 0$ and $a \neq 1$. If base $a = e \approx 2.718$, the exponential function becomes the (natural) exponential function, $f(x) = e^x$. Related to this, as m gets larger, $(1 + \frac{1}{m})^m$ approaches e.

Exponential functions are used in financial formulas. If principal (present value) amount P is invested at interest rate r per year over time t, *simple interest*, I, is $I = Prt$. If P is invested at interest rate r per year, compounded m times per year for t years, *compound amount* is

$$A = P\left(1 + \frac{r}{m}\right)^{mt}.$$

If interest rate r is compounded *continuously*, compound amount after t years is

$$A = Pe^{rt}.$$

Logarithmic functions are related to exponential functions. Assume $a > 0$, $a \neq 1$ and $x > 0$

$$y = \log_a x \quad \text{if and only if} \quad a^y = x \;\; (\text{or } a^y - x = 0)$$

where "$\log_a x$" is read "logarithm of x to the base a". If base $a = e$, the logarithmic function becomes the (natural) logarithmic function, $f(x) = \log_e x = \ln x$; if base $a = 10$, the logarithmic function becomes the (common) logarithmic function, $f(x) = \log_{10} x = \log_{10} x$. For any positive x, y and a, $a \neq 1$, and any real number r,

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a x^r = r \log_a x$

Also, $\log_a a = 1$, $\log_a 1 = 0$ and $\log_a^r = r$. The change-of-base theorem for logarithms

$$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$$

and change-of-base theorem for exponentials is

$$a^x = e^{(\ln a)x}$$

For y_0 amount present at time $t = 0$, let amount present at time t be

$$y = y_0 e^{kt}.$$

If $k > 0$, then k is a *growth constant* and y is an exponential growth function (used in bacterial growth, for example); if $k < 0$, then k is a *decay constant* and y is an exponential decay function (used in radioactive decay, for example). In addition to this *unbounded* model, the *limited growth model* is given by

$$y = L - (L - y_0)e^{kt}$$

where $k < 0$ and L is a limit to growth.

Also, *effective rate for compound interest* is

$$r_E = \left(a + \frac{r}{m}\right)^m - 1$$

which becomes $r_E = e^r - 1$ if interest is compounded continuously.

Chapter 3. The Derivative (lecture notes 5,6)

The *limit* L of function $f(x)$ as x approaches (but does not equal) a (from both sides of a) is written

$$\lim_{x \to a} f(x) = L$$

where a and L are both real numbers and where values of $f(x)$ approach (and perhaps equal) L. The limit of L as x approaches a *does not exist* if

- as x approaches a from both sides, $f(x)$ approaches either positive (denoted $\lim_{x\to a} f(x) = \infty$) or negative infinity (denoted $\lim_{x\to a} f(x) = -\infty$), or
- as x approaches a from one side, $f(x)$ approaches positive infinity, but as x approaches a from the other side $f(x)$ approaches negative infinity or vis-versa,
- $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = M$, where $L \neq M$

If a, A, B are real numbers, f and g are functions and

$$\lim_{x\to a} f(x) = A, \qquad \lim_{x\to a} g(x) = B,$$

then

1. If k is a constant, $\lim_{x\to a} k = k$ and $\lim_{x\to a}[k \cdot f(x)] = k \cdot \lim_{x\to a} f(x) = k \cdot A$
2. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$
3. $\lim_{x\to a}[f(x) \cdot g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)] = A \cdot B$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}, if B \neq 0$
5. If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$

6. For any real k, $\lim_{x\to a}[f(x)]^k = [\lim_{x\to a} f(x)]^k = A^k$, provided limit exists
7. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$
8. For any real number $b > 0$, $\lim_{x\to a} b^{f(x)} = b^{[\lim_{x\to a} f(x)]} = b^A$
9. For any real number b where $0 < b < 1$ or $b > 1$, $\lim_{x\to a}[\log_b f(x)] = \log_b[\lim_{x\to a} f(x)] = \log_b A$, if $A > 0$

Limits *at infinity* for $f(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$, such as $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, determined by

- dividing $p(x)$ and $q(x)$ by *highest power of* $q(x)$ *(not* $p(x)$*!)*
- then, for positive real n, using

$$\lim_{x\to\infty} \frac{1}{x^n} = 0, \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

  where if $x < 0$, x^n does not necessarily always exist (for example, for $n = \frac{1}{2}$) and so limit also does not exist in these cases

Roughly speaking, a function is continuous if its graph can be drawn without lifting the pencil from the paper. A function $f(x)$ is continuous at $x = c$ if

1. $f(c)$ is defined,
2. $\lim_{x\to c} f(x)$ exists,
3. $\lim_{x\to c} f(x) = f(a)$.

If $f(x)$ is not continuous, it is *discontinuous*. A function is continuous on an *open* interval, (a, b), if it is continuous on every x in the interval; a function is continuous on a *closed* interval, $[a, b]$, if it is continuous

- on the open interval (a, b),
- from the right at $x = a$,
- from the left at $x = b$.

Polynomial and exponential functions are continuous for all x; rational functions, $f(x) = \frac{p(x)}{q(x)}$, are continuous for all x where $q(x) > 0$; logarithmic functions, $f(x) = \log_a x$, $a > 0$, $a \neq 1$, are continuous for all $x > 0$; root functions, $f(x) = \sqrt{ax+b}$, $ax + b \geq 0$, are continuous for all x where $ax + b \geq 0$.

The *average rate of change* of $f(x)$ with respect to x as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}.$$

The *instantaneous rate of change* of $f(x)$ at $x = a$ is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{b\to a} \frac{f(b)-f(a)}{b-a},$$

assuming limit exists and where $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(b)-f(a)}{b-a}$ are different versions of the *difference quotient*. These formulas serve as an intermediate step towards understanding the derivative.

Two equivalent definitions of the *derivative* of $f(x)$ at x are

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{b\to x} \frac{f(b)-f(x)}{b-x}$$

if the limit exists, and where *function* $f'(x)$ is read "f-prime of x". Function $f'(x)$ is both the instantaneous rate of change of $y = f(x)$ at x and also the *slope* of the *tangent line* at x. The tangent line to graph of $y = f(x)$ at point $(x_1, f(x_1))$ is

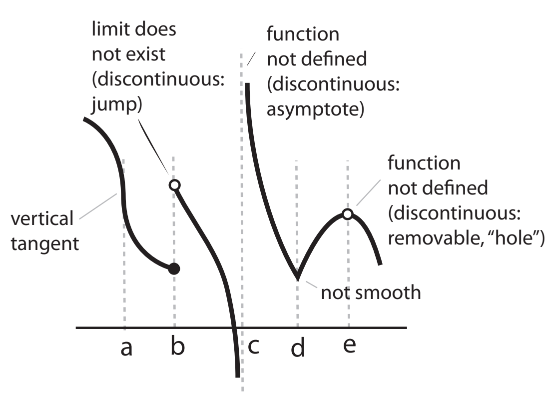
$$y - f(x_1) = f'(x_1)(x - x_1).$$

If $f'(x)$ exists, $f(x)$ is *differentiable* and the steps which produce $f'(x)$ is called *differentiation*. A function f is differentiable if *all* of the following conditions are satisfied,

- f is *continuous*,
- f is *smooth*,
- f does *not* have a vertical tangent line,

and *non*differentiable is any *one* of the following conditions are satisfied,

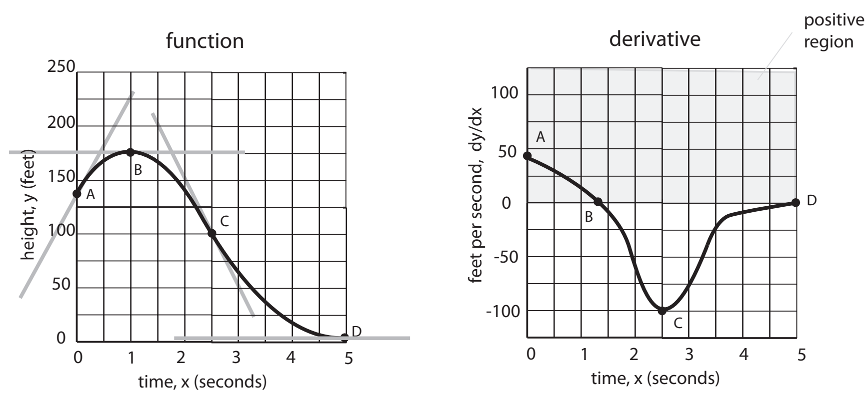
- f is *dis*continuous (there are "jumps", "holes", asymptotes in the function) because a slope cannot be where there is nothing (points b, c and e in Figure);
- f is *not* smooth (there is "sharp corner" in the function) because there are different conflicting slopes (but not *one* slope) at this point (point d in Figure);
- f has a vertical tangent line because the "run" is zero in the rise/run formula for the slope which would make the slope undefined at this point (point a).



Different types of nondifferentiability

Some things to remember when drawing derivatives of functions *graphical*: identify points of function where derivatives

- zero ("peaks and valleys" of a function); for example, at B and D in Figure,
- negative/positive (downward/upwarding sloping parts of a function); for example, positive derivative in region between points A and B, negative in region between points B and D in Figure,
- large (slope of function is either vertical or near-vertical) and also use previous positive/negative information to designate large positive/negative derivative; for example, large negative tangent at C gives minimum derivative in Figure



Drawing graph of a derivative from graph of a function

Chapter 4. Calculating the Derivative (lecture notes 7,8)

Various notations for the derivative of $y = f(x)$ include

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}\left[f(x)\right], \quad D_x[f(x)].$$

Some rules for differentiation include:

- *Constant rule.* Derivative of a constant function, $f(x) = k$, k real, is zero:

$$f'(x) = 0.$$

- *Power rule.* Derivative of $f(x) = x^n$, n real, is

$$f'(x) = nx^{n-1}.$$

- *Constant times function rule.* Derivative of $f(x) = k \cdot g(x)$, k real, $g'(x)$ exists:

$$f'(x) = kg'(x).$$

- *Sum or difference rule.* Derivative of $f(x) = u(x) \pm v(x)$, and $u'(x), v'(x)$ exist:

$$f'(x) = u'(x) \pm v'(x).$$

Product rule: If $f(x) = u(x) \cdot v(x)$, $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x).$$

Quotient rule: If $y = \frac{u(x)}{v(x)}$, $u'(x)$ and $v'(x)$ exist, and $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

One possible *composition* of functions g and f is composed function $g[f(x)]$ whose values are given for all x in the domain of f such that $f(x)$ is in the domain of g. Roughly, composed function $g[f(x)]$ takes"output" of $f(x)$ and uses it as "input" of function $g(x)$, or that $f(x)$ is the inner layer and $g(x)$ is the outer layer and of the function. The *chain rule* is used to find the derivative of the composed function $y = g[f(x)]$, where $y = f(u)$ and $u = g(x)$, and is given by,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'[g(x)] \cdot g'(x)$$

The *exponential* $e \approx 2.71828...$ is defined as

$$\lim_{m \to \infty} \left(1 + \frac{1}{m}\right).$$

The *(natural) exponential function* has the remarkable property it is its own derivative:

$$\frac{d}{dx} e^x = e^x,$$

whereas the derivative of the *exponential function* a^x, $a \neq 1$, is

$$\frac{d}{dx} a^x = (\ln a) a^x.$$

Derivative of *(natural) logarithmic function* is:

$$\frac{d}{dx} \ln |x| = \frac{1}{x} = x^{-1}$$

whereas derivative of the *logarithmic function* is:

$$\frac{d}{dx} [\log_a |x|] = \frac{1}{(\ln a)x} = ((\ln a)x)^{-1}.$$

Chapter 5. Graphs and the Derivative (lecture notes 10,11)

For any two values x_1 and x_2 in an interval,

$$f(x) \text{ is increasing if} f(x_1) < f(x_2) \quad \text{if} \quad x_1 < x_2,$$
$$f(x) \text{ is decreasing if} f(x_1) > f(x_2) \quad \text{if} \quad x_1.$$

Derivatives can be used to determine whether a function is increasing, decreasing or constant on an interval:

$$f(x) \text{ is increasing if} \qquad \text{derivative } f'(x) > 0,$$
$$f(x) \text{ is decreasing if} \qquad \text{derivative } f'(x) < 0,$$
$$f(x) \text{ is constant if} \qquad \text{derivative } f'(x) = 0.$$

A *critical number*, c, is one where $f'(c) = 0$ or $f'(c)$ does not exist; a *critical point* is $(c, f(c))$. After locating the critical number(s), choose test values in each interval between these critical numbers, then calculate the derivatives at the test values to decide whether the function is increasing or decreasing in each given interval. (In general, identify values of the function which are *dis*continuous, so, in addition to critical numbers, also watch for values of the function which are not defined, at vertical asymptotes or singularities ("holes").)

A relative (local) extremum (plural: extrema) ia defined as follows:

$$f(c) \text{ is } \textit{relative (local) maximum} \qquad \text{if } f(x) \leq f(c), \text{ for all } x \text{ in (a,b)}$$
$$f(c) \text{ is } \textit{relative (local) minimum} \qquad \text{if } f(x) \geq f(c), \text{ for all } x \text{ in (a,b)}$$
$$f(c) \text{ is } \textit{relative (local) extrema} \qquad \text{if } c \text{ is either a relative minimum or maximum at } c.$$

If function f has a relative extremum at c, then c is either a critical number or an endpoint. *First derivative test* for locating relative extrema in (a, b):

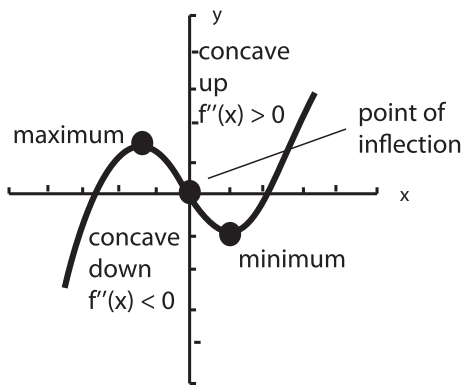
$$f(c) \text{ is } \textit{relative maximum} \qquad \text{if } f'(x) \text{ positive in } (a, c), \text{ negative in } (c, b)$$
$$f(c) \text{ is } \textit{relative minimum} \qquad \text{if } f'(x) \text{ negative in } (a, c), \text{ positive in } (c, b)$$

Although treated in a similar manner, allowances are made for identifying relative extrema for functions with discontinuities.

Notation for *higher* derivatives of $y = f(x)$ include

$$\text{second derivative:} \quad f''(x), \quad , \tfrac{d^2y}{dx^2}, \quad D_x^2[f(x)],$$
$$\text{third derivative:} \quad f'''(x), \quad , \tfrac{d^3y}{dx^3}, \quad D_x^3[f(x)],$$
$$\text{fourth and above, } n\text{th, derivative:} \quad f^{(n)}(x), \quad , \tfrac{d^{(n)}y}{dx^{(n)}}, \quad D_x^{(n)}[f(x)].$$

Second derivative of function can be used to check for both concavity and points of inflection of the graph of function.



Concave up, concave down and point of inflection

In particular, if function f has both derivatives f' and f'' for all x in (a, b), the

$f(x)$ is *concave up* if $f''(x) > 0$, for all x in (a,b)
$f(x)$ is *concave down* if $f''(x) < 0$, for all x in (a,b)

At an *inflection point* of function f, either $f''(x) = 0$ or second derivative does not exist (although the reverse is *not* necessarily true). *Second derivative test* is used to check for relative extrema. Let f'' exist on open interval containing c (except maybe c itself) and let $f'(c) = 0$, then

if $f''(c) > 0$ then $f(c)$ is *relative minimum*
if $f''(c) < 0$ then $f(c)$ is *relative maximum*
if $f''(c) = 0$ or $f''(x)$ does not exist test gives no information, use first derivative test

We combine a number of previous ideas to sketch a graph of a function. First, determine the following properties of the function:

1. *domain*, note restrictions

   cannot divide by 0, or take square root of negative number, or take logarithm of 0 or of a negative number

2. *y-intercept, x-intercept*, if they exist

   y-intercept: let $x = 0$ in $f(x)$, x-intercept: solve $f(x) = 0$ for x

3. *vertical, horizontal, oblique asymptotes*

   vertical asymptote when denominator 0, horizontal asymptote when $x \to \infty$, or $x \to -\infty$

4. *symmetry*

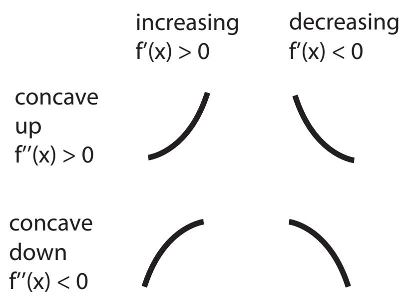
   symmetric about y-axis if $f(-x) - f(x)$; symmetric about origin if $f(-x) = -f(x)$

5. *first derivative test*

   note critical points (when $f'(x) = 0$), relative extrema, in/decreasing sections of function

6. *points of inflection, intervals and concavity*
   note inflection points (when $f''(x) = 0$), concave up/down

Then, plot all points and connect them with a smooth curve, taking into account asymptotes, concavity and in/decreasing sections of function. Check result with a graphing calculator. Commonly recurring shapes are given in the figure; for example, an increasing concave up function is given in upper left corner.



Some important function shapes

Chapter 6. Applications of the Derivative (lecture notes 12)

Function f at c, $f(c)$, in an interval has an

$$\begin{array}{ll} \textit{absolute maximum} & \text{if } f(x) \leq f(c), \\ \textit{absolute minimum} & \text{if } f(x) \geq f(c), \end{array}$$

for all x in the interval. Technique for identifying absolute extrema depends on whether the interval is open or closed. For *closed* intervals, the *extreme value theorem* is used to identify absolute extrema. This theorem says a continuous function f on a closed interval $[a, b]$ will/must have both an absolute maximum and absolute minimum. Consequently, in the closed interval case,

1. evaluate f all critical numbers in (a, b),
2. evaluate f at endpoints a and b of $[a, b]$,
3. largest f is absolute maximum; smallest is absolute minimum.

If f is defined on an *open* interval (a, b), evaluate the *limit* of f as it approaches the endpoints; there is no absolute extrema if the limit is $\pm\infty$. In the special case when there is only *one* critical number c, the *critical point theorem* says for function f defined on (either open or closed) interval I,

$$\begin{array}{ll} \text{if } f \text{ has relative minimum at } x = c & \text{this relative minimum is an absolute minimum,} \\ \text{if } f \text{ has relative maximum at } x = c & \text{this relative maximum is an absolute maximum.} \end{array}$$

Although treated in a similar manner, allowances are made for identifying absolute extrema for functions with discontinuities. A related question of finding the maximum of $g(x) = \frac{f(x)}{x}$ is given by finding x such that $f'(x) = \frac{f(x)}{x}$, in other words, finding the x where slope of tangent, $f'(x)$ *equals* slope of line from origin to point x.

We apply the techniques used in absolute minima and maxima problems to a number of applied topics. Steps to solving applied problems include:

- Determine *function.*

  draw a picture to make things clear if possible

  identify variable to maximize or minimize, express this variable as a function of *one* other variable

- Determine *domain* of function.

  identify if open or closed interval

- Determine *critical values and endpoints.*

- Identify *absolute extrema.*

  check results with calculator

Chapter 7. Integration (lecture notes 13)

- *power rule* $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
- *constant multiple rule* $\int k \cdot f(x) \, dx = k \int f(x) \, dx + C$
- *sum or difference rule* $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
- *exponential functions*
  1. $\int e^{kx} \, dx = \frac{e^{kx}}{k} + C$, $k \neq 0$
  2. $\int a^{kx} \, dx = \frac{a^{kx}}{k(\ln a)} + C$, $a > 0, a \neq 1$
- $\int \frac{1}{x} \, dx = \int x^{-1} \, dx = \int \frac{dx}{x} = \ln |x| + C$

Use boundary conditions to determine the constant of integration, C.

We look at an integration technique called *substitution*, which often simplifies a complicated integration. Roughly, the substitution integration technique is the reverse of the chain rule differentiation technique. We use the following formulas as a basis for the substitution technique, after substituting $u = f(x)$ (and so $du = f'(x)dx$).

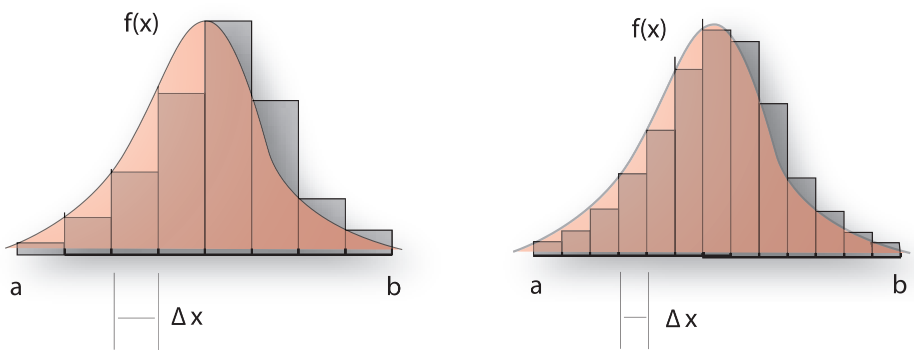
- $\int [f(x)]^n f'(x) \, dx$ becomes $\int u^n \, du = \frac{u^{n+1}}{n+1} + C$, $n \neq 1$
- $\int e^{f(x)} f'(x) \, dx$ becomes $\int e^u \, du = e^u + C$

- $\int \frac{f'(x)}{f(x)}\,dx$ becomes $\int \frac{1}{u}\,du = \int u^{-1}\,du = \ln|u| + C$

Substitution method typically concerned with three cases; chose substitution u to be

- quantity under root or raised to a power
- quantity in denominator
- exponent of e

and allow for constants. We also look at how to deal with fractions in integration. So far, we have looked at *in*definite integrals; now, we turn to *definite* integrals. An *in*definite integral determines the area under a curve; a *definite* integral determines a *specific* area under a curve between a lower bound a and an upper bound b.



Approximating area with sum of rectangles

As shown in the figure, the area under the curve, between points a and b, can be approximated by adding the area of n rectangles and this approximation improves the greater the number of increasingly narrow rectangles. If f is defined on interval $[a, b]$ the definite integral is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where the limit exists, $\Delta x = \frac{b-a}{n}$ and x_i is (somewhere, possibly to the left or to the right) in the ith interval. We look at different ways the rectangles are summed, whether using the left endpoints or from the right endpoints or from the middle endpoints or left and right total areas are averaged. In economic applications, the definite integral is called *total change*.

Chapter 13. The Trigonometric Function (lecture notes 9,14)

Consider an angle with origin (vertex) at the origin of the coordinate system and two *rays* where the *initial side* ray is along the x-axis and *terminal side* ray is at the end of a rotation of angle θ. *Acute*, *right*, *obtuse* and *straight* angles occur when $0^o < \theta < 90^o$, $\theta = 90^o$, $90^o < \theta < 180^o$ and $\theta = 180^o$ respectively. For *radius*, r and *arc (length)*, s, of a circle, *radian measure* of θ is defined as $\frac{s}{r}$; where, notice, if radius of circle is one (1), a *unit* circle, radian measure equals arc length s. An angle can be measured in either *degrees* or *radians*, where

$$1 \text{ radian} = \frac{180^o}{\pi}, \qquad 1^o = \frac{\pi}{180} \text{ radians.}$$

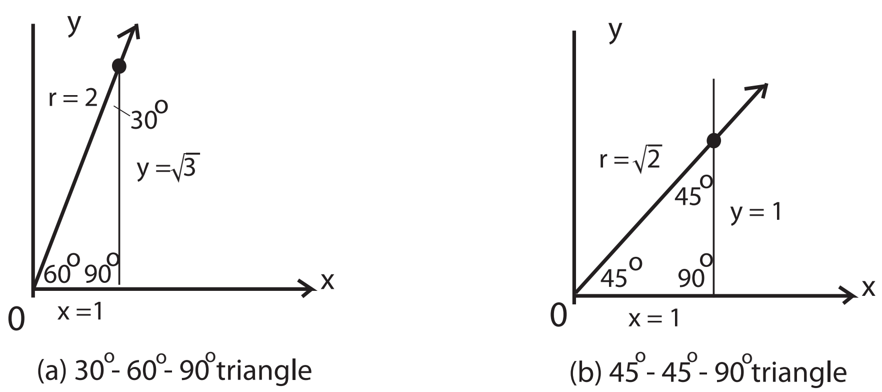
Let r be distance from origin to point (x, y) on terminal side ray. Then

$$\begin{aligned} \sin\theta &= \frac{y}{r} & \csc\theta &= \frac{r}{y}, \quad (y \neq 0) \\ \cos\theta &= \frac{x}{r} & \sec\theta &= \frac{r}{x}, \quad (x \neq 0) \\ \tan\theta = \frac{y}{x}, \quad (x \neq 0) & & \cot\theta &= \frac{x}{y}, \quad (y \neq 0) \end{aligned}$$

where notation t or x can be used instead of θ; for example, $\sin t$ or $\sin x$ could be used instead of $\sin\theta$. Also, some trigonometric identities are:

$$\begin{aligned} \csc\theta &= \frac{1}{\sin\theta} & \sec\theta &= \frac{1}{\cos\theta} & \cot\theta &= \frac{1}{\tan\theta} \\ \tan\theta &= \frac{\sin\theta}{\cos\theta} & \cot\theta &= \frac{\cos\theta}{\sin\theta} & \sin^2\theta + \cos^2\theta &= 1 \end{aligned}$$

Values for trigonometric functions are typically found using a calculator but some values can be found for triangles with special angles given in the Figure. For example, for $30^o - 60^o - 90^o$ triangle, where $60^o = 60 \cdot \left(\frac{\pi}{180}\right) = \frac{\pi}{3}$, so $\sin 60^o = \sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}}{2}$.



Trigonometric functions and special angles

A trigonometric function is *periodic* because it is a function $y = f(x)$ with real number a such that $f(x) = f(x + a)$ for all x; *smallest* a, when the function repeats itself, is the *period* of the function. Periods of both $\sin x$ and $\cos x$ are 2π; their *amplitudes* (half their range ("height") from -1 to 1) are both 1. Furthermore, constants a, b, c, d transform graphs of both $a\sin(bx + c) + d$ and $a\cos(bx + c) + d$ in the following ways:

- amplitude a increases (decreases) "height" of graph for $|a|$ large (small) when $a < 0$, graphs reflected in x-axis ("flipped"),
- constant b (assume $b > 0$) affects period;
  for example, graph of $y = \sin(bx)$ looks like $y = \sin x$ but with period $T = \frac{2\pi}{b}$
  if $0 < b < 1$, period completed more rapidly (shorter period) than when $b = 1$
  if $b > 1$, period completed more slowly (longer period) than when $b = 1$
- horizontal shift c moves graphs *left* ($c > 0$) or *right* ($c < 0$)
- vertical shift d moves graphs up ($d > 0$) or down ($d < 0$)

Phase shift, $\frac{c}{b}$, gives number of units $\sin bx$ or $\cos bx$ are shifted horizontally; for example, if $c = 2\pi$, $b = 1$, then $\sin(x + 2\pi)$ is $\frac{c}{b} = \frac{2\pi}{1} = 2\pi$ units *left* of $\sin x$, whereas if $c = 2\pi$, $b = 2$, then $\sin(2x + 2\pi)$ is $\frac{2\pi}{2} = \pi$ units *left* of $\sin 2x$.

Trigonometric identities:

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x}$$

Sum–difference identities

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\cos(x-y) &= \cos x \cos y + \sin x \sin y \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\sin(x-y) &= \sin x \cos y - \cos x \sin y
\end{aligned}$$

Derivatives of trigonometric functions include:

$$\begin{aligned}
D_x[\sin x] &= \cos x & \qquad D_x[\csc x] &= -\cot x \csc x \\
D_x[\cos x] &= -\sin x & \qquad D_x[\sec x] &= \tan x \sec x \\
D_x[\tan x] &= \sec^2 x & \qquad D_x[\cot x] &= -\csc^2 x
\end{aligned}$$

Integrals of trigonometric functions include:

$$\begin{aligned}
\int \sin x \, dx &= -\cos x + C & \qquad \int \cos x \, dx &= \sin x + C \\
\int \sec^2 x \, dx &= \tan x + C & \qquad \int \csc^2 x \, dx &= -\cot x + C \\
\int \sec x \tan x \, dx &= \sec x + C & \qquad \int \csc x \cot x \, dx &= -\csc x + C \\
\int \tan x \, dx &= -\ln|\cos x| + C & \qquad \int \cot x \, dx &= \ln|\sin x| + C
\end{aligned}$$

Notes	Chapter	Topics	Description
2	1	Graphing Functions	Y = function, GRAPH, WINDOW, TRACE, Zoom
2	1	Graphical Intersection	2nd CALC, Intersection
2	1	Evaluating Functions	Y = function, VAR Y–VAR ENTER ENTER (function) ENTER
4	2	Logarithm Function	logBASE 2 X ENTER ($f(x) = \log_2 x$)
5	3	Numerical Limits	Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 10 ENTER and so on
5	3	Piecewise Functions	Type $y = (-x)(X < 0) + (2)(x = 0)$ into Y_1 =, use 2nd TEST
6	3	Numerical Differentiation	MATH nDeriv(ENTER X ENTER (Function) ENTER (x value) ENTER
8	4	Graphical Differentiation	GRAPH TRACE (x value) 2nd CALC dy/dx
9	13	Trigonometric Functions	SIN, COS, TAN, MODE (either RADIAN or DEGREE)
13	7	Lists	$L_1, \ldots, L_6$
13	7	Summary Statistics	STAT CALC 1–Var Stats L_1
13	7	List of Differences	2nd LIST OPS 7:ΔList)
13	7	Numerical Integration	2nd LIST OPS seq 3X, X, 0.05, 4.95, 0.1) STO 2nd L_2 ENTER
13	7	More Numerical Integration	2nd LIST MATH sum ENTER 2nd L_2) × 0.1 ENTER