Conference on Diophantine m-tuples and related problems

November 13-15

Purdue University North Central

Westville, Indiana

—— PLENARY LECTURES ——

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Alan Filipin

The extension of D(4)-pair $\{\mathbf{k-2, k+2}\}$

In this talk we prove that $D(4)$-triple of the form $\{k-2, k+2, c\}$ where $k \geq 3$ and c are integers has a unique extension to a quadruple with a larger element. In this talk we will present some recent ideas that were used in solving those kind of problems that make our proof much shorter. Some of the ideas are using linear form in two logarithms instead of three and improvement of hyper-geometric method. At the end of the talk we will mention some other results too, that we got using the mentioned methods.

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Yasutsugu Fujita

Diophantine quintuples: Bounds on the elements and the number

A folklore conjecture states that there exists no Diophantine quintuple. It has been known that various kinds of Diophantine triples or pairs cannot be extended to Diophantine quintuples, such as $\{1, 3, 8\}$ by Baker and Davenport or more generally $\{k-1, k+1\}$ with $k \geq 2$ an integer. The first goal of this talk is to give some recent results on the Diophantine pairs that cannot be contained in Diophantine quintuples, one of which is the pair $\{a, b\}$ with $a < b \leq 3a$ (joint work with Cipu). The second half of the talk is devoted to considering the bounds on the number of Diophantine quintuples. In 2004, Dujella showed that there exists no Diophantine sextuple and that there exist only finitely many Diophantine quintuples. Shortly afterward, he gave the explicit upper bound 10^{1930} on the number of Diophantine quintuples. The bound has recently been reduced to $6.8 \cdot 10^{32}$ (joint work with Elsholtz and Filipin) and very recently to 10^{31} by Cipu. We conclude the talk by giving some ideas to get the last two bounds.

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Florian Luca

Diophantine m-tuples with values in linear recurrences

In this talk, we will survey existing results concerning Diophantine m-tuples with values in linear recurrences and present some new results. We shall start by mentioning a result obtained jointly with Fuchs and Szalay in 2008 which states that if $\{u_n\}_{n\geq 1}$ is a binary re- current sequence satisfying certain conditions, then there are at most finitely many triples of positive integers $a < b < c$ such that $ab + 1; ac + 1; bc + 1$ are all members of $\{u_n\}_{n\geq 1}$. The sequences of Fibonacci and Lucas numbers satisfy the conditions of the above theorem, and all Diophantine triples of positive integers with values in the Fibonacci sequence or the Lucas sequence were computed in joint work with Szalay in 2008 and 2009. Later Szalay and his co-authors extended the method to find all Diophantine triples with values in a certain parametric family of binary recurrent sequences which includes the Fibonacci sequence as a particular case. We shall also report on some new results concerning Diophantine quadruples with values in the Tribonacci sequence. This last result is joint with Carlos Alexis Gómez Ruíz and will be part of his Ph.D. dissertation.

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—— CONTRIBUTED TALKS ——

Alejandra Alvarado (with **Jean Joel Delorme**)

On the Diophantine Equation $x^4 + y^4 + z^4 + t^4 = w^2$

To our knowledge, only three parametric solutions to the equation

$$x^4 + y^4 + z^4 + t^4 = w^2$$

are known.

In this talk, we will consider the equation

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2.$$

and we will show that there are infinitely many parametric solutions by finding points on an elliptic curve over $\mathbb{Q}(m)$. This is joint work with Jean Joel Delorme.

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Sadek Bouroubi (with **Ali Debbache**)

An explicit formula and some new identities around balancing numbers

A positive integer n is called by A. Behara and G. K. Panda a balancing number, if there exist a positive integer r, such that

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

G. K. Panda and P. K. Ray defined cobalancing numbers as solutions of the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r).$$

Let $a > 0$ and $b \geq 0$ be coprime integers. If for some positive integers n and r, we have

$$(a+b) + \cdots + (a(n1) + b) = (a(n+1) + b) + \cdots + (a(n+r) + b)$$

then we say that $an + b$ is an (a, b)-balancing number. In this work, we present an explicit formula for balancing, cobalancing, (a, b)-type cobalancing numbers and some new identities.

Keywords: balancing number, cobalancing number, (ab)-type cobalancing number.

Classification MSC2010: 05A16.

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Selin Çenberci

On the soulutions of the Diophantine equation $\mathbf{X^2 + B^m = Y^n}$

Firstly we considered the Diophantine equation $X + q^m = p^n$, where p and q are odd primes satisfying $q^2 + 1 = 2p^2$ and some other additional conditions, and we gave a conjecture by using the equation $X^2 + B^2 = Y^4$ as an analogue of Terai's Conjecture.

In this paper I will give some special cases such that $y = 5 \pmod 8$ or $y2 = Y = 1 \pmod 8$ which holds both our conjecture and Terai's Conjecture.

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Zrinka Franusić

A method of constructing Diophantine quadruples in some number fields

Let R be a commutative ring with unity 1 and $w \in R$. We say that a set of four nonzero distinct elements in R is a *Diophantine quadruple with the property* $D(w)$ if the product of any two distinct elements increased by w is a perfect square. Such sets can be constructed effectively using polynomial formulas. The aim of this is to verify the conjecture that the existence of a Diophantine quadruple with the property $D(w)$ depends on the representability of w as a difference of two squares of elements in some number fields.

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Edray Goins

There exist infinitely many rational Diophantine 6-tuples almost

A set $\{m_1, m_2, \ldots, m_n\}$ of rational numbers is said to be a rational Diophantine n-tuple if $m_i\, m_j + 1$ is a perfect square for $i \neq j$. In the late 1700's, Euler showed that there exist infinitely many rational Diophantine 5-tuples. It is not known whether there exist infinitely many (nontrivial) rational Diophantine 6-tuples, although Gibbs found seven examples in 1999. In this talk, we use properties of the elliptic curve

$$E_k : y^2 = (1 - x^2)\,(1 - k^2\, x^2)$$

to explain how to find an infinite family of nontrivial 6-tuples. We are motivated by Dujella's results from 2001 using properties of elliptic curves. In the process, we find families of elliptic curves having large rank for the torsion subgroup $Z_2 \times Z_4$.

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George Grossman (with **Yifan Zhang**)

On Diophantine triples

In this paper consider the Diophantine triples, $D(n)$-$\{a, b, c\}$; it is shown how the value(s) of $n = n(x, y)$, not necessarily unique, can also be found from solutions of Diophantine equation

$$xy(y - x) + Ax + By = 0,$$

such that A, B depend on a, b, c.

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Bo He (with **Wenquan Wu**)

On Diophantine quintuple conjecture

In this talk, we will show that if $\{a, b, c, d, e\}$ with $a < b < c < d < e$ is a Diophantine quintuple, then $d < 10^{74}$.

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Borka Jadrijević

p-adic version of the index form equations in a parametric family of bicyclic biquadratic fields

In this talk, we deal with the question of existence of primitive integral elements α having index $\mu(\alpha)$ divisible by fixed primes in one parametric family of bicyclic biquadratic fields. This problem is reduced to solving p-adic analogue of the index form equations in a given family of biquadratic fields.

For a (partial) answer on above question we need some additional conditions: an upper bound for the index $\mu(\alpha)$ or an upper bound for the parameter c. We have two different approaches to this problem, depending on given additional conditions.

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István Pink (with **Bo He, Ákos Pintér, Alain Togbé**)

On the Diophantine inequality $|\mathbf{X^2 - cXY^2 + Y^4}| \leq \mathbf{c+2}$

Generalizing some earlier results, we find all the coprime integer solutions of the Diophantine inequality

$$|X^2 - cXY^2 + Y^4| \leq c + 2, \qquad (X, Y) = 1,$$

except when $c \equiv 2 \pmod 4$, in which case we bound the number of integer solutions. Our work is based on the results on the Diophantine equation

$$AX^4 - BY^2 = C,$$

where A, B are positive integers and $C \in \pm\{1, 2, 4\}$.

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John Robertson

Dujella's Unicity Conjecture–Heuristics and Other Topics

The Dujella unicity conjecture is that for any $k > 0$, the equation

$$x^2 - (k^2 + 1)y^2 = k^2$$

has at most one solution with $0 < y < k-1$ and $x > 0$. We show that the Dujella unicity conjecture holds when $k^2 + 1$ is divisible by exactly two distinct odd primes. We discuss heuristics for the "expected" number of counterexamples to the Dujella unicity conjecture.

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Ivan Soldo

$\mathbf{D(-1)}$-triples of the form $\{1, b, c\}$ and their extensibility in the ring $\mathbb{Z}[\sqrt{-t}], t > 0$

We study $D(-1)$-triples of the form $\{1, b, c\}$ in the ring $\mathbb{Z}[\sqrt{-t}]$, $t > 0$, for positive integer b such that b is a prime, twice prime and twice prime squared. We prove that in those cases c has to be an integer. In cases of $b = 2, 5, 10, 17, 26, 37$, or 50 we prove that $D(-1)$-triples of the form $\{1, b, c\}$ cannot be extended to a $D(-1)$-quadruple in the ring $\mathbb{Z}[\sqrt{-t}], t > 0$, except in cases $t \in \{1, 4, 9, 16, 25, 36, 49\}$. Even it is known that in case of $\mathbb{Z}[i]$ there exist infinitely many $D(-1)$-quadruples of the form $\{1, b, c, d\}$, for t=1 and other exceptional cases of t we show that there exist infinitely many $D(-1)$-quadruples of the form $\{1, b, -c, d\}$, $c, d > 0$ in $\mathbb{Z}[\sqrt{-t}]$.

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László Szalay

Linear recurrences $\mathbf{u_n = Au_{n-1} - u_{n-2}}$ and Diophantine tuples

This talk is related to Florian Luca's presentation and gives an illustration how to determine all diophantine triples if a class of binary recurrences is given. More precisely, we show that there is no positive integer triple $\{a, b, c\}$ such that all of

$$ab + 1, \; ac + 1 \quad \text{and} \quad bc + 1$$

are in the sequence $\{u_n\}_{n\geq 0}$ satisfies the recurrence $u_n = Au_{n-1} - u_{n-2}$ with the initial values $u_0 = 0, \; u_1 = 1$, and $A \in \mathbb{N}^+$.

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Huilin Zhu

Jesmanowicz-Terai-Cao-Le conjecture and pure ternary exponential Diophantine equations

In this talk, we will introduce the Jesmanowicz-Terai-Cao-Le Conjecture and some results in pure ternary exponential Diophantine equations. We will discuss further research plan and ideas in this direction.

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Volker Ziegler

On S-Diophantine m-tuples

Let S be a fixed set of primes and let $a_1, \ldots, a_m$ be positive Distinct integers. We call the m-tuple $(a_1, \ldots, a_m)$ S-Diophantine, if for all $i \neq j$ the integers $a_i a_j + 1 = s_{i,j}$ are S-units, i.e. are the product of primes in S. In this talk we consider the question wheter S-Diophantine quadruples exists if $|S| = 2$.

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